Travelling Wave Solutions and Numerical Analysis of the Fisher-KPP Propagation

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Contents

Abstract

This paper aims to put forth a new approach to construct the travelling wave solutions of the partial differential equation put forth by Ronald Fisher, Andrey Kolmogorov, Ivan Petrovsky, and N. Piskunov. In further sections we display the numerical analysis of this propagation that is widely used to model the spatial distribution of population.

1 Introduction

A reaction-diffusion equation looks like the heat equation with a function $f(u)$ added on:

$$u_t = \triangle u + f(u) \tag{1}$$

Since $f(u)$ may be non-linear, explicit solutions cannot usually be found. Whereas the linear wave equation propagates arbitrary solutions at a fixed speed, reaction-diffusion equations may single out certain wave forms and allow only these to propagate without distortion [3]. A typical problem for an equation of this kind investigates the existence, form, and stability of these traveling waves. Such a solution can be written as:

$$u(x,t) = U(z),\ z = x - ct \tag{2}$$

In 1937, Ronald Fisher and, independently, Andrey Kolmogorov, Ivan Petrovsky, and N. Piskunov (KPP) investigated the wavelike spread of advantageous genes in a population. Fisher's equation was one-dimensional and had a specific "logistic" reaction term:

$$u_t = Du_{xx} + ru(1 - \frac{u}{K}) \tag{3}$$

Fisher proposed this equation as a model of diffusion of a species in a one-dimensional habitat, where D is the diffusion constant, r is the growth rate of the species, and K is the carrying capacity. A dimensionless version of the equation takes the form of the one dimensional, semi-linear parabolic partial differential equation with the dimensional parameters being diffusivity, the proliferation rate and the carrying capacity density:

$$u_t - u_{xx} = f(u),\ t,x \in \mathbb{R} \tag{4}$$

While the original cause of developing this equation was to activate the gene distribution within a population, the Fisher-KPP model and its extensions support travelling wave solutions that are successfully used to design numerous invasive phenomena with applications in biology, ecology, and combustion theory. The equation has been described by Andrey

Nicolaevich Kolmogorov (1903-1987) as the natural population propagation, mass transfer, the processes of chemical reaction, and heat [2]. It is made use of in genetics to define the population distribution with growth management as well as to indicate the density of individuals or particles. In cell biology, the spatial spreading of invasive cell populations has been modelled using the Fisher-KPP model and its extensions for a range of applications including in vitro cell biology experiments and in vivo malignant spreading. Other areas of application include combustion theory and bushfire invasion. Some of the extensions of the Fisher-KPP model involve working with different geometries, such as inward and outward spreading in geometries with and without radial symmetry.

The multiple variations of the Fisher-KPP model include considering models with nonlinear diffusivity; incorporating different nonlinear transport mechanisms; models of multiple invading subpopulations; and multi-dimensional models incorporating anisotropy. However, in the context of the above, this paper aims to consider the Fisher-KPP equation in 2D spatial domain:

$$u_t - k\triangle u = \alpha u(1-u) \tag{5}$$

The Fisher-KPP equation can be interpreted as a model of population dynamics in a one-dimensional environment, where particles simply replace other particles and, therefore, their density remains bounded. All fluctuations of the microscopic model have been neglected to obtain the Fisher-KPP equation.

2 Mathematical Interpretation of Fisher-KPP Propagation

The mathematical study of reaction-diffusion equations began in the 1930s. Fisher and Kolmogorov, Petrovsky and Piskunov were interested in wave propagation in population genetics modeled by the homogeneous equation:

$$\frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} = ru(1-u) \tag{6}$$

The reaction part of the equation can be understood by taking the predator-prey model where $\dot{u} = auv - bu$ acts as the predator and $\dot{v} = cv - duv$ acts as prey. We now enslave the prey to the predator ($u = \alpha v$). In this way, the set of equations adopts a form we just studied. Moreover, with the change of variable $P^* = \frac{b}{\gamma}P$, we get:

$$\frac{\delta P}{\delta t} = \gamma P(1-P) + D\frac{\delta^2 P}{\delta x^2} \tag{7}$$

Here, the * has been dropped for clarity. By ignoring the diffusion term, it is possible to identify the logistic differential equation, i.e. the continuous realization of the logistic map: $x_{n+1} = ax_n(1 - x_n)$. Both analog models yield to the conclusion that the underlying mechanism is that of the growth of a self-limiting reproductive population. In fact, the equation including diffusion was first suggested by Ronald Fisher in 1937 as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population.

The Fisher-KPP model gives rise to travelling wave-like solutions that do not allow the solution to go extinct and will propagate indefinitely on a semi-infinite domain. In other words, it is a self-similar wave front which moves in both space and time.

By taking a generalized form of the propagation into account, we agree to the variables which can occur with the population lives in the habitat, i.e. the growth rates differ in space and time. This could lead to various possibilities, including 1. A particular location is better suited to the inhabitants; 2. The habitat encountered regular (nearly periodic) change; and 3. The population is affected by inherent periodic change which differs based on season.

What makes Fisher-KPP interesting is that many of the equations that describe a front $h(x,t)$ where a stable state invades an unstable state have the same properties as the Fisher-KPP equation. The properties shared by all the fronts in the class of the Fisher-KPP equation include A. There is an unstable constant solution and a stable constant solution; we always write the equation so that $h(x,t) = 0$ is unstable and $h(x,t) = 1$ is stable, and B. There exist positive travelling wave solutions $h(x,t) = v(x - vt)$ if and only if $v >= v_c$ for some critical velocity v_c.

In the Fisher-KPP equation, $v_c = 2$.

3 Travelling Wave Solutions

A travelling wave is a wave that advances in a particular direction, with the addition of retaining a fixed shape. Moreover, a travelling wave is associated to having a constant velocity throughout its course of propagation. Such waves are observed in many areas of science, like in combustion, which may occur as a result of a chemical reaction. In mathematical biology, the impulses that are apparent in nerve fibres are represented as travelling waves. Also, in conservation laws associated to problems in fluid dynamics, shock profiles are characterised as travelling waves. Furthermore, the structures present in solid mechanics are typically modelled as standing waves. Hence, it is important to determine the dynamics of such solutions.

A travelling wave solution is obtained upon solving a model that corresponds to a system. Generally, these models take the forms of partial differential equations (PDEs), where the dynamics of the systems are comprehended upon solving for solutions. These travelling wave solutions are expressed as $u(x, t) = U(z)$, where $z = x - ct$. Here, the spatial and time domains are represented as x and t, with the velocity of the wave given as c.

If $c = 0$, the resulting wave is named a stationary wave. Such waves do not propagate, and are typically observed when inducing a fixed boundary. For a travelling wave that approaches constant states given by $U(-\infty) = u_l$ and $U(\infty) = u_r$, with $u_l = u_r$, we have what we call a wave front. However, if the constant states are equal with $u_l = u_r$, the corresponding wave is known as a pulse wave. If a wave exhibits periodicity with $U(z + F) = U(z)$, where $F > 0$, the wave is called a spatially periodic wave.

4 Construction of Travelling Wave Solutions to Fisher-KPP Equation

4.1 Approach 1

In this approach proposed by Jonkhout [4], we make use of the Fisher-KPP equation in its dimensionless form (4). This has been derived by scaling time to the growth factor, distance to the diffusion length, and population size to the maximum population. Therefore, only values of u between zero and one are relevant.

A traveling wave solution is a solution which satisfies $u(x, t) = \phi(x - ct)$ for some function $\phi : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. The function ϕ is the wave profile and c, the wave velocity.

Now, this approach for construction of travelling wave solutions to the Fisher-KPP equation includes introducing the co-moving frame write $\xi = x - ct$. Upon substituting $\phi(x - ct)$ for u, the left-hand side of the equation becomes:

$$u_t(\xi) = \frac{d\phi}{d\xi}(\xi)\frac{d\xi}{dt}(\xi) = -c\phi'(\xi) \tag{8}$$

In this case, the right hand side becomes:

$$u_x x + u(1 - u) = \phi''(\xi) + \phi(\xi)(1 - \phi(\xi)) \tag{9}$$

Combining both these, we procure the ordinary differential equation:

$$\phi'' + c\phi' + \phi(1 - \phi) = 0 \tag{10}$$

However, while we can find explicit solutions of the equation for specific values of C, the exact solutions cannot be easily determined, since the equation is non-linear. Therefore, to study the behaviour of such equations, we write is as a two-dimensional system of first order equations by setting $\psi = \phi'$ and then moving on to determine the character of the fixed points by computing the Jacobian matrix J of this system.

As explained by Jonkhout, after calculating the characteristic equations and the eigenvalues at $(0,0)$ and $(0,1)$, we consider two cases for c separately: $0 < c < 2$ and $c > 2$.

In the case $0 < c < 2$, they derive that there are no orbits $(\psi(\xi), \phi(\xi))$ such that $\psi(x - ct)$ is a travelling wave that is relevant in the context of population dynamics. However, in the case $c > 2$, we make use of triangle OAB in a heteroclinic orbit, where O is the origin, A is $(1,0)$ and B is the point $(1, -b)$. We prove that there is an orbit that leaves $(1,0)$ and enters OAB and that no orbit can leave the triangle in forward direction. As a consequence of the monotone convergence theorem, we receive a limit in this case. By the Poincare-Bendixon theorem, this limit must be the first coordinate of a fixed point. The limit is not equal to one, and, since there are only two fixed points, it is equal to zero.

4.2 Approach 2

Recall that the Fisher-KPP equation in 1-D is given by:

$$F_t - kF_{xx} = \alpha F(1 - F), \tag{11}$$

where, k and α are constants. We now look for solutions $f(t)$ such that $F(t, x) = f(x - ct)$ for some constant C (speed). Using the chain rule, we have:

$$F_t(t, x) = \frac{d}{dt} f(x - ct) = -cf' \tag{12}$$

$$F_x x(t, x) = \frac{d^2}{dx^2} f(x - ct) = f'' \tag{13}$$

Therefore, f solves the following ordinary differential equation:

$$-cf' - kf'' = \alpha f(1 - f) \tag{14}$$

$$\lim_{x \to -\infty} f(x) = 1 \qquad \lim_{x \to +\infty} f(x) = 1$$

Main Idea

We know that function f lies from $\mathbb{R}$ to $[0,1]$. By approximation, let us assume that $a > 0$ is a big fixed number. Let f_a be the solution of:

$$-c_a f_a' - k f_a'' = \alpha(f(1-f)) \tag{15}$$

$$f_a(-a) = 1 \quad f_a(a) = 0$$

Here, f_a is a function $[-a, a] \to [0, 1]$.

The system above has more than 1 solution for different values of c_a. In fact, it has a solution for every c_a. In order to specify the speed uniquely, we must impose an additional normalisation limitation $f_a(0) = \frac{1}{2}$.

Theorem 1 *For all a the system stated above has a solution, that satisfies bounds*

$$\lim_{x\to-\infty} f(x) = 1 \textit{ and} \lim_{x\to+\infty} f(x) = 1. \tag{16}$$

Moreover, it is a subsequence a_n such that

$$\lim_{x\to+\infty} f_{a_n}(x) = f(x) \textit{ and} \lim_{x\to+\infty} ca_n(x) = \sqrt{2k\alpha} \tag{17}$$

5 Numerical Solution of the Fisher-KPP Equation

The Reduced Differential Transform Method (RDTM) was primarily introduced by Keskin and Oturanc [5] in 2009, which is an extension of the differential tranform method, first submitted by Zhou. RDTM is an iterative procedure for obtaining Taylor series solution of differential equations. The solutions achieved by the RDTM is an infinite power series for initial value problems, which can be, in turn expressed in a closed form, the exact solution.

According to the basic properties of the RDTM, we can find the transformed form of the Fisher–KPP equation by starting with initial approximations $u(x, 0) = u_0(x) = U_0(x)$ and moving on by obtaining the rest of the components. We then take the inverse transformations, thereby obtaining an n-terms approximate solution:

$$u(x,t) = \lim_{x\to+\infty} u(x,t) = \lim_{x\to+\infty} \left(\sum_{k=0}^{n} U_k(x) \right) \tag{18}$$

We can then use this to calculate the numerical results [6].

6 Conclusion

Through the means of this paper, we have explored two approaches that can be used to construct travelling wave solutions for the Fisher-KPP propagation. Using the Reduced Differential Transform Method, we also derived an equation that can be used to calculate numerical solutions to the equation.

7 Scope for Further Research

A possible research area related to this paper would be the addition of a Line of Fast Diffusion [1] and studying the impacts of the same on a non-linear partial differential equation such as the Fisher-KPP propagation, from a numerical standpoint. In this model the two-dimensional environment will include a line on which fast diffusion takes place while reproduction and usual diffusion only occur outside this line. For low diffusion, the line has no effect, whereas, past a threshold, the line enhances global diffusion in the plane and the propagation is directed by diffusion on the line.

References

[1] Henri Berestycki, Jean-Michel Roquejoffre, and Luca Rossi. The influence of a line with fast diffusion on fisher-kpp propagation. *Communications in Mathematical Sciences*, 2012.

[2] Maud El-Hachem, Scott W. McCue, Wang Jin, Yihong Du, and Matthew J. Simpson. Revisiting the fisher-kpp equation to interpret the spreading-extinction dichotomy. *bioRxiv*, 2019.

[3] Francois Hamel and Lenya Ryzhik. On the nonlocal fisher-kpp equation: Steady states, spreading speed and global bounds. *Nonlinearity*, 27, 07 2013.

[4] C.J.H. Jonkhout. Traveling wave solutions of reaction-diffusion equations in population dynamics. 2016.

[5] Oturanç G. Keskin Y. *International Journal of Nonlinear Sciences and Numerical Simulation*, 10:741, 2009.

[6] Rostam K. Saeed and Andam A. Mustafa. Numerical solution of fisher–kpp equation by using reduced differential transform method. *AIP Conference Proceedings*, 1888(1):020045, 2017.